HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH ONLY PERIODIC SOLUTIONS

BY

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To the Memory of Eri Jabotinsky

ABSTRACT

We determine all third order homogeneous linear differential equations with periodic coefficients and only periodic solutions. The method extends to nth order equations. As an application, we show that the Laguerre-Forsyth canonical form cannot be used for global investigations in projective differential geometry.

1. Recently, I determined all differential equations (Hill equations)

$$x'' + p(t)x = 0, \qquad p(t) = p(t + \pi), \tag{1}$$

all whose solutions are periodic [1]. The procedure was suggested by the fact that (1) is essentially the Frenet equation of a plane curve in unimodular centroaffine differential geometry. The method used for (1) does not extend to higher orders. In the present paper, we indicate a geometric method for the construction of all nth order homogeneous linear differential equations with periodic coefficients and only periodic solutions. Because of the complication of the formulae, we work the solution out only for $n = 3$. As an application, we discuss the canonical form of the equation of plane projective differential geometry used by Wilczynski [4] and Lane [2]. These equations are based on a lift of the curves of the projective plane into affine three-space. It turns out that, in general, the canonical lift of a closed curve is not closed.

2. We consider a space curve

$$x(t) = (x_1(t), x_2(t), x_3(t))$$

whose coordinate functions are three linear independent solutions of a third order equation

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$$x''' + P_2 x'' + P_1 x' + P_0 x = 0, \tag{2}$$

with continuous coefficients P_i. The Wronskian determinant of the space curve $[x, x', x'']$, is different from zero and this property is true for all regular parameters on the curve, not just the parameter t used in (2). We write

$$x(t) = r(t)c(t),$$

where $r(t) = |x(t)|$ and $c(t) = x(t)/|x(t)|$ is the unit vector in direction x. Let s be the arclength of the spherical curve c, c_2 the unit tangent vector and $c_3 = c \times c_2$ the tangent normal. The Frenet equations of the frame (c, c_2, c_3) are

$$\frac{d}{ds}\begin{bmatrix} c \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} c \\ c_2 \\ c_3 \end{bmatrix}.$$

We indicate differentiation with respect to s by a dot. From

$$\begin{aligned} \dot{x} &= \dot{r}c + rc_2, \\ \ddot{x} &= (\ddot{r} - r)c + 2\dot{r}c_2 + r\gamma c_3 \end{aligned}$$

it follows that $[x, \dot{x}, \ddot{x}] = r^2\gamma \neq 0$: *The projection into the unit sphere of a solution curve of an equation* (2) *is a spherical curve of nowhere vanishing geodesic curvature* γ.

A simple computation then gives

$$x^{\cdots} + Q_2 x^{\cdot\cdot} + Q_1 x^{\cdot} + Q_0 x = 0$$

where
$$\begin{aligned} Q_2 &= -(r\gamma)^{\cdot}/r\gamma, \\ Q_1 &= 1 + \gamma^2 + 2\frac{\dot{r}\dot{\gamma}}{r\gamma} + 2\frac{\dot{r}^2}{r^2} - 3\frac{\ddot{r}}{r}, \\ Q_0 &= -(1+\gamma^2)\dot{r} + \frac{1}{r}(3\dot{r}\dddot{r} + 3\dot{r} - \ddot{r}) + \frac{(r\gamma)^{\cdot}}{r^2\gamma}(2\dot{r}^2 + r - \ddot{r}). \end{aligned}$$

Hence, we see that *every equation* (2) *with only periodic solutions is equivalent to an equation* (3) *with* γ *the geodesic curvature of a closed spherical curve,* i.e., a function

$$\gamma = \cos\theta(\phi^{\cdot}\theta^{\cdot\cdot} - \phi^{\cdot\cdot}\theta^{\cdot}) + \sin\theta(1 + \theta^{\cdot 2})\phi^{\cdot},$$

where $\theta(s)$ and $\phi(s)$ are *periodic* functions of period $L = s(\pi)$ linked by

$$\theta^{\cdot 2} + \phi^{\cdot 2}\cos^2\theta = 1.$$

3. The solution given above is unsatisfactory in that the parameter s has no obvious meaning for the equation (2). However, we may remark that

$$\int_0^L Q_2\,ds = 0$$

since $r\gamma \neq 0$ and that, therefore, every equation (3) is equivalent to an equation

$$x''' + P_1 x' + P_0 x = 0 \tag{4}$$

with *periodic* coefficients P_1, P_0 and this in a canonical way. Since now the Wronskian is a constant, the corresponding parameter t is a multiple of the unimodular centroaffine parameter (the integral invariant, unique up to an additive constant, for the linear action of the special linear group.) In particular, no generality is lost if we assume $[x, x', x''] = 1$. We put $\alpha = \gamma^{-1/3}$, then

$$u = dt/ds = \alpha/r, \quad x' = r'c + \alpha c_2,$$

and

$$x''' = Ac + Bc_2,$$

where

$$A = r''' - 3\alpha\alpha' r^{-1},$$

$$B = \frac{1}{r^2}(2r''r\alpha - \alpha^3 + rr'\alpha' - r'^2\alpha + \alpha'' - \alpha^{-3}). \tag{5}$$

This means that

$$P_1 = \alpha^{-1}B, \quad P_0 = r^{-1}(A - Br'). \tag{6}$$

The differential equation (4) *has all its solutions periodic if and only if its coefficients depend by* (6) *and* (5) *on the geodesic curvature of a regular space curve without inflections.*

4. We may also investigate the equations

$$y''' + Q(t)y = 0, \quad Q(t) = Q(t+T). \tag{7}$$

This equation has been taken by Wilczynski [4] and Lane [2] as basis of a treatment of plane projective differential geometry. A curve in the real projective plane is a map

$$x: I \to P^2$$

of the interval I into the projective plane P^2. The three-dimensional cartesian space minus the origin is mapped onto P^2 by

$$\pi(x_1, x_2, x_3) = (x_1 : x_2 : x_3).$$

A space curve

$$y: I \to R^2 - 0$$

is a *lift* of the projective curve x if

$$x(t) = \pi y(t).$$

All lifts have the same projection into the unit sphere.

Among all the differentiable lifts of a differentiable $x(t)$, we look for those that satisfy

$$[y, y', y''] = 1, \; [y, y'', y'''] = 0.$$

Then t is the unimodular centroaffine parameter for y. By the preceding section, the problem is to find $u(t)$ such that $B(t) = 0$. If we use u to eliminate r, we obtain

$$2u\ddot{u} - 3\dot{u}^2 = u^2 F(\alpha),$$

$$F(\alpha) = 1 + \alpha^{-2} - 3\ddot{\alpha}/\alpha.$$

The substitution $u = v^{-2}$ gives

$$\ddot{v} + \tfrac{1}{4}F(\alpha)v = 0. \tag{8}$$

It follows from Lyapounov theory [3] that a non-constant periodic or semi-periodic solution of (8) must vanish somewhere (the number of zeros in $0 \leqq t < L$ being the index of the eigenvalue $\lambda = 1$ of the problem $v(L) = \pm v(0)$ for $\ddot{v} + (\lambda F/4)v = 0$.) Hence, *the lift of a closed projective curve for which the unimodular centroaffine equation is in the Laguerre-Forsyth normal form* (7) *is not a regular curve unless* $F(\alpha) = 0$.

In addition, the Laguerre-Forsyth lift is not closed in general. For instance, for constant $\gamma = \tan \theta_0$, the curve c is a parallel circle of length $L = 2\pi \cos \theta$. The radius

$$r = \cos^{-2}\left(\frac{s}{2\cos\theta_0}\right) + \theta_0.$$

becomes infinite for $s = (\pi - 2s_0)\cos\theta_0$.

References

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